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Multiplayer Bargaining with Delayed Agreement

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Abstract

The best known equilibrium strategies of multiplayer bargaining define that the agreement is established at the first moment. In this paper two new subgame perfect Nash equilibria strategies are proposed, one in which the agreement moment is delayed for $T > 1$ periods and one other in which the bargaining proposals proceed endlessly.

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1 Introduction

In unanimity multiplayer bargaining there are three or more players and a divisible good to be shared among them. The division is obtained by the following procedure: at each period a player proposes a division and the other players vote simultaneously in favor or against it. If at least one player votes against the proposal, the game goes on to another round, with another player proposing a division and a new suffrage taking place. At each round, the good in question loses value by a discount factor $\delta \in (0,1)$. The game ends when a proposal is accepted by all, the division is made accordingly.

The best known multiplayer bargaining result is that all points in the unitary simplex are subgame perfect Nash equilibria (SPNE) outcomes of the game, meaning that all divisions can be agreed upon in equilibrium. Herrero (1985) created an ingenious strategy valid for 3 players when $\delta > 1/2$. Haller (1986) noted that an equilibrium for all divisions could be extended to $\delta \leq 1/2$. But in both these strategies the agreement is established in the first period. However, equilibria with a different agreement period can be implemented, namely one with *agreement at time* $T > 1$ and other with *no agreement* at all. Although

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the existence of equilibrium strategies with agreement later than the first period has been mentioned in the literature, see for example Chatterjee and Sabourian (1999) or Serrano (2008), to our knowledge they have never been described or proved. The purpose of this paper is to address this gap.

The following section presents the model and the notation. In section 3 Haller's equilibrium strategy is presented, in section 4 the strategies with delayed agreement are defined. In section 5 we conclude.

2 Notation and Classical Equilibria

The set of players is $I = \{1, 2, 3\}$[1]. At period $t \in \mathbb{N}$ one proposal is made by the proponent at t the player $i(t)$. Propositions are made in a cyclic order and $i(t) : T \to I$ determines the proponent $i(t) = \{i \in I : i = t \ (mod\ 3)\}$. The proposal is a point of the unitary simplex $\Delta = \{(x_1, x_2, x_3) : \sum_{i=1}^{3} x_i \leq 1, x_i \geq 0\}$.

Player's response to the proposal is an action taken on $\{0, 1\}$; with a_j^t, for $j \neq i(t)$, the action of j at t, being 0 if j rejects the proposition received, and 1 if the player accepts it. So, if $i \neq i(t)$, $a_i^t \in \{0, 1\}$; if $i = i(t)$, $a_i^t \in \Delta$. For the sake of simplicity define the set of actions available for i at t by

$$A_i^t = \begin{cases} \{0, 1\} \text{ if } i = i(t) \\ \Delta \text{ if } i \neq i(t) \end{cases}$$

The vector of all actions at moment t is $a^t = (a_1^t, a_2^t, a_3^t)$ and the space of all actions at t is $A^t = A_1^t A_2^t A_3^t$. We will also need the set of actions that do not establish an agreement $\tilde{A}^t = \{a \in A^t : (a_j, a_k) \neq (1, 1) \text{ and } j, k \neq i(t)\}$

A history is a sequence of actions that can either end after or before the proposition is done, and a distinction between these two cases is necessary. For any $t \in \mathbb{N}$ a $(t, 2)$-*history* is a history with t complete stages, that is t propositions and voting. A $(t, 1)$-*history* is a history with $t - 1$ complete stages plus one proposition.

The space of $(t, 2)$-stage histories is, for $t \geq 2$, $H^{t,2} = \left(\prod_{k=1}^{t-1} \tilde{A}^k\right) \times A^t$; for $t = 1$, the $(1, 2)$-stage histories is $H^{1,2} = A^1$. $H^{0,2}$ stands for $\emptyset$ the unique 0-stage history. The space of all $(t, 1)$-stage histories is $H^{t,1} = \left(\prod_{k=1}^{t-1} \tilde{A}^k\right) \times \Delta$. The set of all histories is $H = \bigcup_{t=1}^{\infty} (H^{t,1} \cup H^{t,2}) \cup H^{0,2}$. The proposal at t in history $h \in H$ is $h^{t,1}$ and the responses are $h^{t,2}$, the actions taken in period t are $h^t = (h^{t,1}, h^{t,2}) \in A^t$. The length of a history, $\tau(h)$ is a function from the set of histories into the stage moment $\tau : H \mapsto \mathbb{N}_0 \times \{1, 2\}$, so $\tau(h) = (t, k)$, with $t \in \mathbb{N}_0$ being the period of the history, and $k \in \{1, 2\}$ whether the

[1]We will use the case with 3 players to simplify the proofs, although the results are valid for $n > 3$ players.

voting has already taken place $k = 2$ or not $k = 1$. $t(h)$ is the period of history h, so $\tau(h) = (t(h), k)$ and $i(h) = i(t(h))$ is the proponent at h. The part of history h until stage (t, k), with $\tau(h) \geq (t, k)$, is $h^{|t,k}$. h^+ and h^- are, respectively, the history h plus one more stage or without the last stage, and it will be used only when the marginal actions are obvious from the context. It is assumed that each player has perfect recall. Thus at stage (t, k) each player knows $h^{|t,k}$. $(h, \bar{h})$ is the history composed by h followed by $\bar{h}$.

A *pure strategy for player i* is a function $s_i : H \to A_i^{t(h^+)}$ mapping histories into actions. The set of player i's pure strategies is denoted by S_i, and $S = S_1 \times S_2 \times S_3$ is the joint pure strategy space. For a strategy $s \in S$ and $h \in H$ the induced play by s at h is denoted by $s|h$, with $(s|h)(\bar{h}) = s(h, \bar{h})$. Every combination of pure strategies $s \in S$ induces a path $\varpi_s(h)$ after the history h, $\varpi_s(h) = \Big\{s(h), s\big(h, s(h)\big), s\Big(h, s(h), s\big(h, s(h)\big)\Big), \dots\Big\}$.

A strategy s induces a division $d(s)$ as well as a moment $t(s)$ in which the agreed division occurs. The moment $t(s)$ is when all players accept a proposition, with the played history $\bar{h} = \varpi_s(H^{0,2})$, $t(s) = t(\bar{h})$ and the division is $d(s) = \bar{h}^{t(s),1}$. If there is no agreement, by convention, $t(s) = +\infty$ and $d(s) = \overline{0}$. The payoff for a given strategy is $\Pi_i(s) = \delta^{t(s)} d_i(s)$.

3 Classical Equilibria in Multiplayer Bargaining

In this section we will present the equilibrium defined by Haller (1986). Herrero was the first[2] to prove that all points in Δ are equilibrium outcomes when $\delta > 1/2$. Later Haller noted that the equilibria could be extended to any δ. The equilibrium concept used is naturally the Subgame Perfect Nash Equilibrium that we hereby define.

Definition 1. *Subgame Perfect Nash Equilibrium*
$s \in S$ is an SPNE if $\Pi_i(s|h) \geq \Pi_i(s'_i, s_{-i}|h)$ $\forall h \in H_i, \forall i \in I$ and $\forall s'_i \in S_i$.

Where H_i is the set of histories where player i has to decide, $H_i = \{h \in H : \tau(h) = (t, 1), i(t) \neq i \text{ or } \tau(h) = (t, 2), i(t + 1) = i\}$ The utility function, with $h = \varpi_s(H^{0,2})$, can be written in the form $\Pi_i(s) = \sum_{k=1}^{t(h)} \delta^{k-1} (h_1^k h_2^k h_3^k)_i$, with $h^k = (h_1^k, h_2^k, h_3^k)$ where $h_1^k h_2^k h_3^k$ is the vector in Δ with the payments at k. $(h_1^k h_2^k h_3^k)_i$ is the instant payment for player i at k, it is either zero or the value of the agreed division at k, $h_{i(k)}^k$. It is relatively straightforward to see that if two strategies share the same future path for a long period their actualized payment will be similar. Therefore the utility function is continuous at infinity and the one shot deviation principle is valid (see Fudenberg and Tirole (1991, theorem 4.2, pag 110)). To prove that a given strategy is an SPNE we only need to look for alternative strategies

[2] Although Shaked never published his results it is attributed to him the creation of such strategies, see, for example, Sutton (1986) or Osborne and Rubinstein (1990)

which are different on one information set. For this purpose we define the one shot deviation strategy

Definition 2. *One Shot Deviation Strategies (OSD)*
The set of OSD *from* s_i *at* $h \in H_i$ *is* $OSD(s_i, h) = \{s'_i \in S_i : s'_i(h) \neq s_i(h)$ *and* $s'_i(h') = s_i(h'), \forall h' \in H_i \setminus h\}$.

Haller's strategy works as a finite automata, where the state tracks if any player has deviated from the planned actions and induces punishment for that player. $r(h)$ is the state at each stage history, thus $r(h) : H \to E$ with $E = \{e^0, e^1, e^2, e^3\}$ the set of states. In this strategy the division to be proposed after history h only depends on the state $r(h)$, for this reason we use the same symbol for the state and the division associated with it. e^0 is any point in Δ, e^i is the division in which player i receives everything, $e^1_i = 1$, and both other players receive zero, $e^i_k = 0$. At the initial moment $H^{0,2}$ the state is $r(H^{0,2}) = e^0$ and it changes only after the proposal (and before the replies), thus for $\tau(h) = (t, 2)$, $r(h) = r(h^-)$. For $\tau(h) = (t, 1)$, and considering $h = (h^-, h^{t,1})$

$$r(h) = \begin{cases} r(h^-) & \text{if } h^{t,1} = r(h^-) \\ e^{i(t+1)} & \text{if } h^{t,1} \neq r(h^-) \end{cases}$$

If the player proposes a division determined by the state $h^{t,1} = r(h^-)$ then the state in the next period does not change. If he proposes a different division the state changes to $e^{i(t+1)}$, where player $i(t+1)$ gets everything and the proponent $i(t)$ is penalized. As it is common in punishment schemes not only the deviator, but also other players are penalized (see for example the folk theorem, Fudenberg and Tirole (1991, theorem 5.1, pag 152)). Now we will present the equilibrium strategy.

Definition 3. *Haller Equilibrium Strategy*
For h *such that* $\tau(h) = (t-1, 2)$ *the proposition always equals the state* $s_{i(t)}(h) = r(h)$. *For* $\tau(h) = (t, 2)$ *replier* $j \neq i(h)$ *accept the proposition only if it is equal to the state*

$$s_j(h) = \begin{cases} 1 & \textit{if } h^{t,1} = r(h) \\ 0 & \textit{if } h^{t,1} \neq r(h) \end{cases}$$

Table 1: **Haller's Strategy**

	State	e^j
Player i	Proposal	e^j
	Accept p	$p = e^j$

Theorem 4. *All divisions are an SPNE outcome of Haller's strategy.*

Proof. s is Haller's strategy with $r(H^{0,2}) = e^0$ for any, but fixed, $e^0 \in \Delta$. We will prove that there is no history $h \in H_i$ after which the player can change its strategy to $s'_i \in OSD(s_i, h)$ and improve his payment. Let us start by noting that due to $r(h) = r(h^-)$, for $\tau(h) = (t, 2)$, the actions $h^{t,2}$ have no influence in the state, whatever the responses the state does not change.

For $i = i(t)$ and $\tau(h) = (t-1, 2)$. If players follow s, i proposes $r(h)$ and all others accept, $\Pi_i(s|h) = r_i(h)$. If $s'_i \in OSD(s_i, h)$ then $h^{t,1} = s'_i(h) \neq s_i(h) = r(h)$, i made a different proposal. The state changes to $e^{i(t+1)}$. If $h^{t,1} = e^{i(t+1)}$ then repliers j, k accept and $\Pi_i(s'_i, s_{-i}|h) = e_i^{i(t+1)} = 0$. If $h^{t,1} \neq e^{i(t+1)}$ repliers j, k reject and i's payoff is $\Pi_i(s'_i, s_{-i}|h) = \delta\Pi_i(s'_i, s_{-i}|h^+) = \delta\Pi_i(s|h^+) = \delta e_i^{i(t+1)} = 0$. Clearly $\Pi_i(s'_i, s_{-i}|h) \leq \Pi_i(s|h)$ for any $s'_i \in OSD(s_i, h)$, the proponent has no advantage in altering his strategy.

For $j \neq i(t)$ and $\tau(h) = (t, 1)$ we have two possibilities for the player not to act like in s, either to accept a proposal different from $r(h)$ or to reject the proposal of $r(h)$. When $h^{t,1} = r(h)$, if s is played the proposition is accepted and $\Pi_j(s|h) = r_j(h)$. If $s'_j \in OSD(s_j, h)$, j should refuse the proposition, $s'_j(h) = 0$, we can define $h^{t,2} = \big(s'_j(h), s_k(h)\big) = (0, 1)$ and $h^+ = (h, h^{t,2})$. As the proposition was made according to s the state did not change, so $r(h^+) = r(h)$. j's refusal delays the agreement one period, because after h^+ all players follow s and the agreement is $r(h^+) = r(h)$. $\Pi_j(s'_j, s_{-j}|h) = \delta\Pi_j(s'_j, s_{-j}|h^+) = \delta\Pi_j(s|h^+) = \delta r_j(h^+) = \delta r_j(h)$, and we conclude that $\Pi(s'_j, s_{-j}|h) \leq \Pi_j(s|h)$. In the case the proposition was not equal to the state, $h^{t,1} \neq r(h)$. If $j \neq i(t)$ follows s the proposal is refused, the state is $r(h^+) = r(h) = e^{i(t+1)}$, where $h^+ = (h, (0, 0))$, and $\Pi_j(s|h) = \delta\Pi_j(s|h^+) = \delta e_j^{i(t+1)}$. If j follows $s'_j \in OSD(s_j, h)$ accepting the proposition, $s'_j(h) = 1$. The proposal will still be declined by the other player. There will be no change in state caused by j response, and $r(\bar{h}^+) = e^{i(t+1)}$, with $\bar{h}^+ = (h, (1, 0))$. $\Pi_j(s'_j, s_{-j}|h) = \delta\Pi_j(s|h^+) = \delta e_j^{i(t+1)} = \delta\Pi_j(s|h^+) = \Pi_j(s|h)$. Player j does not improve its payment by changing strategy. □

4 Delayed Agreements

It is of note that under Haller's strategy all divisions, even Pareto dominated, are division outcomes, however agreement is always reached at $T = 1$. It is also possible to obtain an SPNE strategy s for which the agreement is reached later, $t(s) > 1$, through an adaptation of Haller's strategies.

Theorem 5. *All divisions are an SPNE outcome with agreement established at any period $T \in \mathbb{N}$.*

Proof. For any $e^0 \in \Delta$ and $T \geq 2$, $r(h)$ defines again the state, but the set of states is now $E = \{\delta^{T+1}e^0, e^0, e^1, e^2, e^3\}$. For the initial history the state is $r(H^{0,2}) = \delta^{T+1}e^0$, when $t = t(h) > 0$ we have

$$r(h) = \begin{cases} r(h^-) & \text{if } h^{t,1} = r(h^-) \text{ and } t \neq T-1 \\ e^0 & \text{if } h^{t,1} = r(h^-) \text{ and } t = T-1 \\ e^{i(t+1)} & \text{if } h^{t,1} \neq r(h^-) \end{cases}$$

Again the state does not change at the voting period, when $\tau(h) = (t, 2)$ $r(h) = r(h^-)$. When $\tau(h) = (t-1, 2)$ the strategy, like Haller's, is to propose a division equal to the state, $s_{i(h)}(h) = r(h)$. When $\tau(h) = (t, 1)$ replier $j \neq i(h)$ follows

$$s_j(h) = \begin{cases} 1 & \text{if } h^{t,1} = r(h) \text{ and } t \geq T \\ 0 & \text{if } h^{t,1} \neq r(h) \text{or } t < T \end{cases}$$

We need to prove two distinct results, first that $e(s) = T$ and $d(s) = e^0$; second, that s is an SPNE. The first one is relatively straightforward, define $\bar{h} = \varpi_s(H^{0,2})$ as the history path when s was played. For all repliers $j \neq i(t)$ and all partial histories $\bar{h}^{|t,1}$ with $t < T$, $s_j(\bar{h}^{|t,1}) = 0$, then the time of agreement must be $t(s) \geq T$. If s is always played, at $(T-2, 2)$ the state is $r(\bar{h}^{|T-2,2}) = \delta^{T+1}e^0$; then at $(T-1, 1)$ the player $i(T-1)$ proposes the division $r(\bar{h}^{|T-2,2})$ and according to the definition of $r(h)$ the state changes to e^0; at $(T-1, 2)$ both repliers reject the proposition; at $(T, 1)$ proposition will be e^0, $s_{i(T)}(\bar{h}^{|T-1,2}) = r(\bar{h}^{|T-1,2}) = e^0$; then both repliers $j \neq i(T)$ accept $s_j(\bar{h}^{|T,1}) = 1$ so $\min_{j \in I \setminus i(T)} s_j(\bar{h}^{|T,1}) = 1$. The agreement is established at $t(s) = T$ and the division reached is $d(s) = e^0$.

To prove s is an SPNE. If a history h has $t(h) \geq T$ and $r(h^{|T-1,2}) = e^0$ the strategy is identical to Haller's so it respects SPNE condition. If there was some deviation on the propositions and the strategy entered in a punishment scheme, meaning that $r(h) \in \{e^1, e^2, e^3\}$ then it would again replicate the punishment scheme of Haller's strategy, which we already proved to be an SPNE. To conclude that the strategy is an SPNE we only need to establish that s is the best option for histories with $t(h) < T$ and in which the proponents did not deviate, thus histories with $r(h) = \delta^{T+1}e^0$.

When $\tau(h) = (t-1, 2)$ the payment for $i(t) = i$ of following s is $\Pi_i(s|h) = \delta^{T-t}e_i^0$. If i proposes something different $s_i'(h) \neq \delta^{T+1}e^0$, the state changes to $r(h^+) = e^{i(t+1)}$. If $s_i'(h) \neq e^{i(t+1)}$ both repliers reject the proposition and $\Pi_i(s'|h) = \delta\Pi_{i)}(s|h^+) = \delta r(h^+)_i = \delta e_i^{i(t+1)} = 0$. If $s_i'(h) = e^{i(t+1)}$ repliers accept the proposal and $\Pi_i(s'|h) = e_i^{i(t+1)} = 0$. There is no gain for the proponent $i(t)$ when he changes to an $OSD(s_{i(t),h})$.

For $j \neq i(t)$, $\tau(h) = (t, 1)$ with $t < T$ and $h^{t,1} = r(h^-) = \delta^i T + 1e^0$ the agreement will be reached in $T - t$ periods, so $\Pi_j(s|h) = \delta^{T-t} e^0_j$. When j plays $s'_j \in OSD(s_j, h)$, $s'_j(h) = 1$ player j contradicts s accepting the proposal, but replier $k \neq j$ still rejects it and so, with $h^+ = \big(h, (0, 1)\big)$, $\Pi_j(s'_j, s_{-j}|h) = \delta\Pi_j(s|h^+) = \delta\big[\delta^{T-(t+1)} e^0\big] = \Pi_j(s|h)$. Replier j does not improve his payment by changing strategy. □

If the players are not interested in bargaining, and they always propose everything for themselves and reject anything less, then we have another atypical SPNE outcome, where an agreement is never established. The next theorem will prove the existence of such an equilibrium.

Theorem 6. *There is a SPNE strategy $s \in S$ in which no division is agreed upon.*

Proof. For the proponent $i = i(h)$ the strategy is $s_i(h) = e^i$. For the repliers $j \neq i(h)$ with $\tau(h) = (t, 1)$, $s_j(h) = \begin{cases} 1 & \text{if } h^{t,1} = e_j \\ 0 & \text{if } h^{t,1}_j \neq e_j \end{cases}$. It is clear that no agreement can be reached in finite time, the replier j only accepts e^j and replier $k \neq j$, e^k, therefore they will never accept the same proposal, so $\forall h \in H$, $\Pi_i(s|h) = 0$ and $t(s) = \infty$. We still need to prove that s is an SPNE.

When $\tau(h) = (t-1, 2)$, whatever the proposal $s'_{i(t)}(h)$ it will always be rejected by one of the repliers. For $i(t)$ the payment does not increase by using $s'_i \in OSD(s_i, h)$. For any proposition $\bar{h}^{t,1} = s'_i(h)$, $\Pi_i(s'_i, s_{-i}|h) = \delta\Pi_i(s|h, \bar{h}) = 0$.

When $\tau(h) = (t, 1)$, the replier $j \neq i(t)$ cannot improve his payment. If under s he rejected the proposal $s_j(h) = 0$, on the alternative strategy $s'_j \in OSD(s_j, h)$ he accepts it $s'_j(h) = 1$. The proposal at t was $h^{t,1}$ and the payment of j is: $h^{t,1}_j$ if k accepted the proposition; and, is $\delta\Pi_j(s|h^+) = 0$ if k rejected it, with $h^+ = (h, (1, 0))$. So j's payment is $\Pi_j(s'_j, s_{-j}|h) = s_k(h)h^{t,1}_j + (1 - s_k(h))\,\delta\Pi_j(s|h^+) = s_k(h)h^{t,1}_j$. But $s_k(h) = 1$, only when $h^{t,1} = e^k$, meaning that $h^{t,1}_j = e^k_j = 0$, and $\Pi_j(s'_j, s_{-j}|h) = 0$.

If under s the replier accepted the proposal, $s'_j(h) = 0$ but nothing really changes, the game goes to the next round and players will again try to get everything for themselves. So $\Pi_j(s'_j, s_{-j}|h) = \delta\Pi_j(s|h^+) = 0$ the change of reply does not improve the replier's payoff.

□

5 Conclusion

This paper develops two new equilibrium strategies for the multiplayer bargaining game, both based on the strategy presented in Haller (1986), in which the date of the agreement is not the initial moment.

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