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A Constructive Proof of the Nash Bargaining Solution

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Abstract

This paper offers a constructive proof of the Nash Bargaining solution. We start by proving that Nashs solution is representable based on its continuity. This property along with the linearity of the choice function will then allow us to identify the function representing Nashs bargaining choice. Finally, supported on the result for two players, we will generalize it to n-players.

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1 Introduction

The purpose of this paper is to present a constructive proof of the Nash bargaining solution. While Nash (1950) defined axioms that a two-player bargaining should respect and presented a solution, it is not clear from his paper why was that solution chosen or how it came about. We will arrive to the result without guesses on the shape of the bargaining solution, thus constructively bringing the initial axioms and the final solution together.

Peters and Wakker (1991) showed that, under certain conditions, a (bargaining) choice function can be the result of the ordering of well-behaved preferences. If it is

Pareto optimal and continuous there is an equivalence between the choice function being independent of irrelevant alternatives and being representable. That is, if the choice is continuous, Pareto optimal and independent of irrelevant alternatives then exists a function f, that can be interpreted as a social utility function, such that on the set S the choice $c(S) = \arg\max_{\boldsymbol{s}\in S} f(\boldsymbol{s})$.

In order to prove that Nash's solution is representable we will start by demonstrate the continuity of the solution, as Pareto optimality and independence of irrelevant alternatives are two of Nash's axioms. In this process it will also be proved that the function f must be quasi concave. And this property together with the linearity of the choice function will allow us to discover that $f(x, y) = xy$. Finally, supported on the result for two players, we will generalize it to n-players.

In the next section we present the notation and definitions that we will use throughout the text. Then in section 3 we will prove that the Nash Bargaining solution is representable. In section 4 we will then arrive at the function representing the Nash's bargaining choice, after which we will conclude.

2 Notation and Definitions

As most definitions and axioms we will common no justification or intuition will be provided. A vector in $\mathbb{R}^2_+$ will be denoted by a bold letter usually $\boldsymbol{x}$ and its coordinates by $\boldsymbol{x} = (x, y)$. The set of compact and convex sets of $\mathbb{R}^2_+$ is $\mathbb{S}$. For a set $S \in \mathbb{S}$, the ideal value of S for the first player is $S^1 = \max\{x : \exists y \in \mathbb{R}, (x, y) \in S\}$, S^2 the ideal value for player 2. $\mathbb{S}^+$ is the set of the compact and convex subsets of $\mathbb{R}^2_+$ with $S^1S^2 > 0$. A set $S \subset \mathbb{R}^2_+$ is *comprehensive* if $\boldsymbol{x} \in S$ then $\boldsymbol{x'} \in S$ for any $\boldsymbol{x'} \leq \boldsymbol{x}$. The *comprehensive hull* of a set $S \in \mathbb{S}$ is *comp*$(S) = \left\{\boldsymbol{x'} : \boldsymbol{x'} \leq \boldsymbol{x}, \text{ for any } \boldsymbol{x} \in S\right\}$. For a comprehensive $S \in \mathbb{S}^+$ the function $g_S : [0, S^1] \to [0, S^2]$ defines the maximum value of y when the first coordinate has value x, so $(x, y) \in S$ if and only if $y \leq g_S(x)$. The convexity of S implies that g_s is concave. The *convex hull* of S, *ch*(S) is the smallest convex set that contains S. A set is *symmetric* if $(x, y) \in S$ implies $(y, x) \in S$. An *affine transformation* of $\boldsymbol{x} = (x, y) \in \mathbb{R}^2_+$, for $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2_+$ and $\boldsymbol{\beta} = (\beta_1, \beta_2) \in \mathbb{R}^2_+$, is

$\boldsymbol{\beta} + \boldsymbol{\alpha x} = \big(\beta_1 + \alpha_1 x, \beta_2 + \alpha_2 x\big)$. An affine transformation of a set S is $\boldsymbol{\beta} + \boldsymbol{\alpha} S = \big\{\boldsymbol{\beta} + \boldsymbol{\alpha x} : \boldsymbol{x} \in S\big\}$. In the next section we need to use the affine transformation of S that sends the point $\boldsymbol{x} \in S$ into $\tilde{\boldsymbol{x}} \in \mathbb{R}^2_+$ intensively, this will be denoted by $S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}$. The proportion factor $\boldsymbol{\alpha}$ is $\boldsymbol{\alpha} = \frac{\tilde{\boldsymbol{x}}}{\boldsymbol{x}}$, then $S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})} = \boldsymbol{\alpha} S = \frac{\tilde{\boldsymbol{x}}}{\boldsymbol{x}} S = \left(\frac{\tilde{x}}{x}, \frac{\tilde{y}}{y}\right) S$.

The bargaining problem is defined for pairs $(S, \boldsymbol{d})$, in which S is convex and compact set of possible utilities for the players, and it exists an $\boldsymbol{x} \in S$ such that $\boldsymbol{x} \gg \boldsymbol{d}$. We will normalize the disagreement point to $\boldsymbol{d} = \boldsymbol{0}$, this can be done without loss of generality because of the *affine transformation* axiom we will just define. Therefore, a bargaining game will, from now on, be defined just on the sets $S \in \mathbb{S}^+$. The Nash bargaining solution is a function that to each bargaining problem S picks a utility division $c(S) \in S$, and respects the following axioms: *Pareto Optimality(PO)*, for $S \in \mathbb{S}^+, \nexists \boldsymbol{x} \in S \setminus c(S) : \boldsymbol{x} \geq c(S)$; *Independence of Irrelevant Alternatives(IIA)* if $S, S' \in \mathbb{S}^+$ with $S' \subseteq S$ and $c(S) \in S'$ then $c(S) = c(S')$; *Symmetry (Sym)*, for symmetric $S \in \mathbb{S}^+$, $c(S)_1 = c(S)_2$; *Affine Transformations(AT)*, for $S \in \mathbb{S}^+$ and $\boldsymbol{\alpha} \in \mathbb{R}^2_+$ the bargaining choice verifies $c\big(\boldsymbol{\alpha} S\big) = \boldsymbol{\alpha} c(S)$.

3 The Bargaining Choice is Representable

The proof of the continuity of the choice function will be done by *contradiction*, assuming that there is a convergent sequence of sets $\{S_k\}$, $S_k \to S$, such that the bargaining solution is not continuous $c(S_k) \nrightarrow c(S)$. [1] If this is case, there is a sequence of convergent comprehensive sets $comp(S_k) \to comp(S)$ such that $c\big(comp(S_k)\big) \nrightarrow c\big(comp(S)\big)$, because $c\big(comp(X)\big) = c(X)$ by *PO* and *IIA*. Hence, continuity of c can be studied through comprehensive sets and in this section, even if not clearly mentioned, sets are comprehensive.

We will prove that if the solution c was not continuous, there would be a set S'

[1] The distance between two points is $d(\boldsymbol{x}, \boldsymbol{x'}) = \max\big\{|x - x'|, |y - y'|\big\}$; the distance from a set to point is $d(\boldsymbol{x}, S') = \inf_{\boldsymbol{x'} \in S'} d(\boldsymbol{x}, \boldsymbol{x'})$; and the Hausdorff distance between two sets is

$$d(S, S') = \max\big\{\sup_{\boldsymbol{x} \in S} d(\boldsymbol{x}, S'), \sup_{\boldsymbol{x'} \in S} d(\boldsymbol{x'}, S)\big\}$$

.

with the bargaining solution $c(S')$ belonging to the interior of some S_k, being therefore worse than $c(S_k)$; and $c(S_k)$ belonging to the interior of S' and so worse than $c(S')$, thus creating a contradiction with *IIA*. The set S', that will be used to show this contradiction, is an affine transformation of S, one that changes the point $\boldsymbol{x^*} = c(S)$ to a point $\tilde{\boldsymbol{x}}$ in the interior of S, so $S' = S_{(\boldsymbol{x*},\tilde{\boldsymbol{x}})}$. The next lemma will prove that a set S' in the stated conditions exists.

Lemma 1. *For all $\boldsymbol{x}, \boldsymbol{x'} \in S$ with $\boldsymbol{x} \gg \boldsymbol{0}$ and $g_S(x) \neq g_S(x')$, then an $\tilde{\boldsymbol{x}} \in int(S)$ such that $\boldsymbol{x'} \in int\big(S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}\big)$ exist.*

Proof. If $\tilde{x} \in \big(\min\{x, x'\}, \max\{x, x'\}\big)$ then $x' < \tilde{x}/xS^1$: $S^1/x \geq 1$, if $x' = \min\{x', x\}$ then $x' < \tilde{x} \leq \tilde{x}(S^1/x)$; if $x' = \max\{x, x'\}$, $x < \tilde{x} < x'$, then $\frac{\tilde{x}}{x} > 1$, and $x' \leq S^1 < \frac{\tilde{x}}{x}S^1$.

The frontier of the set $\boldsymbol{\alpha}S$ at x' is $g_{\boldsymbol{\alpha}S}(x') = \alpha_2 g_S\big(x'/\alpha_1\big)$, when $\boldsymbol{\alpha} = \tilde{\boldsymbol{x}}/\boldsymbol{x} = (\tilde{x}/x, \tilde{y}/y)$, $\boldsymbol{\alpha}S = S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}$, and $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') = (\tilde{y}/y)g_S\big((x/\tilde{x})x'\big) = (\tilde{y}/y)g_S(\bar{x})$, $\bar{x} = (x/\tilde{x})x'$. If $\tilde{y} = g_S(\tilde{x})$, because $y \leq g_S(x)$, $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') \geq \big(g_S(\tilde{x})/g_S(x)\big)g_S(\bar{x})$ taking logarithms and considering $w = \log g_S$, we get $\log\big(g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x')\big) \geq w(\tilde{x}) - w(x) + w(\bar{x})$. The function w is always a non-increasing and strictly concave function, the logarithm of a non-increasing and concave function is non-increasing and strictly concave. As $\min\{x, x'\} < \tilde{x} < \max\{x, x'\}$ there is a $0 < \theta < 1$ such that $\tilde{x} = x^{\theta}x'^{1-\theta}$ and $\bar{x} = (x/\tilde{x})x' = x^{1-\theta}x'^{\theta}$. Using Jensen's inequality $x^{\theta}x'^{1-\theta} \leq \theta x + (1-\theta)x'$ and as the function w is non increasing $w(\tilde{x}) = w(x^{\theta}x'^{1-\theta}) \geq w\big(\theta x + (1-\theta)x'\big) > \theta w(x) + (1-\theta)w(x')$, the last inequality is derived from $g_S(x) \neq g_S(x')$ and strict concavity of w. Applying this reasoning to the entire equation $\log\big(g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x')\big) \geq w\big(x^{\theta}x'^{1-\theta}\big) - w(x) + w\big(x^{1-\theta}x'^{\theta}\big) > w(x')$. Taking exponentials back we prove that

$$g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') = \frac{g_S(\tilde{x})}{y} g_S(\frac{x}{\tilde{x}}x') > g_S(x') \tag{1}$$

If instead of $\tilde{y} = g_S(\tilde{x})$ we chose a value of $\tilde{y}$ sufficiently close to $g_S(\tilde{x})$ the inequality is preserved. We proved that for $\tilde{x} \in \big(\min\{x, x'\}, \max\{x, x'\}\big)$ and $\tilde{y} < g_S(\tilde{x})$, $x' < (\tilde{x}/x)S^1 = S^1_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}$ and $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') = (\tilde{y}/y)g_S((x/\tilde{x})x') > g_S(x')$, so $\boldsymbol{x'} = (x', g_S(x')) \in \text{int}(S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})})$ for $\tilde{\boldsymbol{x}} \in \text{int}(S)$ □

The next result involves almost no derivation, however it is essential for later use, and for this reason, it has a lemma of its own.

Lemma 2. *$\forall \boldsymbol{x}, \boldsymbol{x'} \in S$ with $\boldsymbol{x} \gg \boldsymbol{0}$ and $g_S(x) \neq g_S(x')$ for $\tilde{\boldsymbol{x}} \in S$ such that $\tilde{x} \in \big(\min\{x, x'\}, \max\{x, x'\}\big)$ and $\tilde{y} = g_S(\tilde{x})$ then $\big(x', g_S(x')\big) \in S_{(\boldsymbol{x}, \tilde{\boldsymbol{x}})}$.*

Proof. $x' \leq (\tilde{x}/x)S^1$, equation (1) insures that $g_{S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}}(x') > g_S(x')$, and with these conditions we derive that $\big(x', g_S(x')\big) \in S_{(\boldsymbol{x},\tilde{\boldsymbol{x}})}$. □

The choice of any set attributes to each player a strictly positive payoff. If this were not the case and $C(S)_1 = 0$ due to the non increasing frontier of the comprehensive set S and *PO*, $C(S) = (0, S^2)$. With $\boldsymbol{\alpha} = (1, S^1/S^2)$ then $c(\boldsymbol{\alpha}S) = (0, S^1)$. The set $\Delta = ch\big\{(0,0), (0,S^1), (S^1,0)\big\} \subseteq \boldsymbol{\alpha}S$ and by *IIA* $c(\Delta) = c(\boldsymbol{\alpha}S) = (0, S^1)$, which is in contradiction with the *sym* axiom, as Δ is symmetric.

Lemma 3. *: If $S^i > 0$ then $c(S)_i > 0$ with $i = 1, 2$.*

An immediate and simple implication of the *IIA* axiom is that two sets with different bargaining choices cannot simultaneously contain the other's set choice. If $c(S) \in S'$, $c(S) \in S \cap S'$, by *IIA* $c(S' \cap S) = c(S)$, using the same argument if $c(S') \in S$ then $c(S' \cap S) = c(S')$, and we get a contradiction if $c(S') \neq c(S)$.

Lemma 4. *: $S, S' \in \mathbb{S}^+$, $c(S) \neq c(S')$ and $c(S) \in S'$, then $c(S') \notin S$.*

We have now reunited the conditions to prove the main result of this section, that the choice function must be continuous.

Theorem 1. *The function c is continuous on $\mathbb{S}^+$.*

Proof. Suppose c is not continuous, then a convergent sequence of sets S_k, $S_k \to S$exists, with the bargaining choice $c(S_k) = \boldsymbol{x_k}$ not convergent to $c(S) = \boldsymbol{x^*}$. We start by requiring that $\{\boldsymbol{x_k}\}_{k=1}^{\infty}$ is convergent to the point $\boldsymbol{x'} \in \mathbb{R}^2_+$. Without loss of generality we can also assume that $c(S) = (x^*, y^*) = (x^*, x^*)$. If $x^* \neq y^*$, as $S^1 > 0$ and $S^2 > 0$, lemma (3) insures $\boldsymbol{x^*} \gg \boldsymbol{0}$, with $\boldsymbol{\alpha} = (1, \frac{x*}{y*})$, $\boldsymbol{\alpha}S_k \to \boldsymbol{\alpha}S$, and $c(\boldsymbol{\alpha}S_k) = \boldsymbol{\alpha}c(S_k) \to \boldsymbol{\alpha x'} \neq \boldsymbol{\alpha x^*} = \boldsymbol{\alpha}c(S)$, with $\boldsymbol{\alpha}c(S) = (x^*, x^*)$, and we have a sequence in the desired conditions. We will divide the proof of this theorem in two cases, one in which one coordinate of $\boldsymbol{x'}$ is equal to x^*, and the other where both coordinates are different.

Case 1: $x^* = x'$ or $x^* = y'$. Without loss of generality assume $x^* = x'$. To prove that exists $\bar{\boldsymbol{x}}_{\boldsymbol{k}} = (x_k, x_k) \in S_k$, that for large k are better than $\boldsymbol{x_k}$ we will start by noticing that there is a sequence of $(d_k, d_k) \in S_k$ which converges to $\boldsymbol{x^*}$. From $S_k \to S$ and the continuity maximum function , it can be easily deduced that $d_k = \max\{s : (s,s) \in S_k\} \to \max\{s : (s,s) \in S\} = x^*$. If $\boldsymbol{x_k} = (x_k, y_k)$, with $\bar{x}_k = \frac{x_k+y_k}{2}$, the point $\bar{\boldsymbol{x}}_{\boldsymbol{k}} = (\bar{x}_k, \bar{x}_k)$ belong to S_k for large k: by hypothesis $\{\boldsymbol{x_k}\}_{k=1}^{\infty}$ is a convergent sequence to $\boldsymbol{x'}$. $\boldsymbol{x^*} \neq \boldsymbol{x'}$, and by *PO* $y' < y^* = x^*$, $\bar{x}_k \to (x'+y')/2 < (x^*+y^*)/2 = x^*$. We know $d_k \to x^*$, so, for large k, $\bar{x}_k < d_k$, $(d_k, d_k) \in S_k$ and due to comprehensibility of S_k, $\bar{\boldsymbol{x}}_{\boldsymbol{k}} \in S_k$

We found that $\bar{\boldsymbol{x}}_{\boldsymbol{k}} \in S_k$ and $c(S_k) = \boldsymbol{x_k}$, if there is a set A_k with $c(A_k) = \bar{\boldsymbol{x}}_{\boldsymbol{k}}$ and $\boldsymbol{x_k} \in A_k$, we contradict lemma (4). The symmetric $A_k = ch\{(0,0), (x_k, y_k), (y_k, x_k)\}$ by *Sym* must have $c(A_k)_1 = c(A_k)_2$, by *PO* $c(A_k) = \bar{\boldsymbol{x}}_{\boldsymbol{k}}$. But $c(A_k) \in S_k$ for large k, and by construction of A_k, $c(S_k) = \boldsymbol{x_k} \in A_k$, $\bar{\boldsymbol{x}}_{\boldsymbol{k}} \neq \boldsymbol{x_k}$, (remember $\bar{x}_k = \bar{y}_k$ but $x_k \neq y_k$ for large k because $x' \neq y'$), and we get a contradiction with lemma(4).

Case 2 In which $x^* \neq x'$ and $x^* \neq y'$. If we prove that exists $\tilde{\boldsymbol{x}} \in S$, such that for at least one $k \in \mathbb{N}$, $\boldsymbol{x_k} \in S_{(\boldsymbol{x^*}, \tilde{\boldsymbol{x}})}$ and $\tilde{\boldsymbol{x}} \in S_k$ we contradict lemma (4) again, because $c\big(S_{(\boldsymbol{x^*}, \tilde{\boldsymbol{x}})}\big) = \tilde{\boldsymbol{x}}$ and $c(S_k) = \boldsymbol{x_k}$. As $S_i > 0$ and lemma (1) is applicable, $\exists \tilde{\boldsymbol{x}} \in \text{int}(S)$ such that $\boldsymbol{x'} \in \text{int}\big(S_{(\boldsymbol{x^*}, \tilde{\boldsymbol{x}})}\big)$, therefore, as $\boldsymbol{x_k} \to \boldsymbol{x'}$, $\boldsymbol{x_k} \in \text{int}\big(S_{(\boldsymbol{x^*}, \tilde{\boldsymbol{x}})}\big)$ for large k. Because $\tilde{\boldsymbol{x}} \in \text{int}(S)$ and $S_k \to S$, then $\tilde{\boldsymbol{x}} \in S_k$ for large k. The contradiction is obtained, and c must be continuous.

As $S_k \to S$, defining the compact rectangle $R = \big\{\boldsymbol{s} : \boldsymbol{0} \leq \boldsymbol{s} \leq \big(S^1+1, S^2+1\big)\big\}$, $S_k \subset R$ for large values of k. If we drop the initial assumption of convergence of $\{\boldsymbol{x_k}\}_{k=1}^{\infty} \subset R$ then there are (at least) two subsequences converging to different values. However, as we just saw, any converging subsequence $\boldsymbol{x_{k_i}}$ must converge to $\boldsymbol{x^*}$, hence it is impossible to have two subsequences converging to a value that is not $\boldsymbol{x^*}$. □

As c is continuous in $\mathbb{S}^+$, *PO* and *IIA* by Peters and Wakker (1991)[corollary 5.7]

$c(S)$ is representable and maximizes a real valued function.

Corollary 1. *$c(S)$ maximizes a real valued function f on $S \in \mathbb{S}$.*

4 Nash's Bargaining Solution

Thus far we discovered that the choice function c is representable. In this section we will deduce the shape of the function f that represents the choice. Based on lemma (2) we start by deriving that the function must be quasi concave. Then joining quasi concavity with the axiom AT on the sets defined by lines we discover the function f.

Theorem 2. *If f is such that $c(S) = \arg\max_{\boldsymbol{x} \in S} f(\boldsymbol{x})$ then f is strictly quasiconcave.*

Proof. The function f is strictly quasi concave if for any $\boldsymbol{x_0}, \boldsymbol{x_1}$ and any $\alpha \in (0,1)$, $\boldsymbol{x_\alpha} = \alpha\boldsymbol{x_0} + (1-\alpha)\boldsymbol{x_1}$ we have that $f(\boldsymbol{x_\alpha}) > \max\{f(\boldsymbol{x_0}), f(\boldsymbol{x_1})\}$. We will first prove the intermediate result that for any line L with negative slope exists $x_2 \in [0, L^1]$ such that the function $t(x) = f\big(x, g_L(x)\big)$ is increasing for $x \in [0, x_2]$ and decreasing for $x \in [x_2, L^1]$. The x_2 in question is such that $c(L) = \big(x_2, g_L(x_2)\big)$.

In lemma (2) we proved that when $\tilde{x} \in \Big(\min\{x, x'\}, \max\{x, x'\}\Big)$, $\tilde{y} = g_S(\tilde{x})$ and $g_S(x) \neq g_S(x')$ then $\big(x', g_S(x')\big) \in S_{(\boldsymbol{x}, \tilde{\boldsymbol{x}})}$. Replacing $x, x', \tilde{x}$ by x_2, x_0, x_1, if $x_1 \in (x_0, x_2)$ we obtain that $\boldsymbol{x_0} \in L_{(\boldsymbol{x_2}, \boldsymbol{x_1})}$, where $\boldsymbol{x_i} = (x_i, g_L(x_i))$ for $i = 0, 1, 2$. $c(L_{(\boldsymbol{x_2}, \boldsymbol{x_1})}) = \boldsymbol{x_1}$, then $t(x_1) = f(\boldsymbol{x_1}) > f(\boldsymbol{x_0}) = t(x_0)$. This result is valid for all $x_0 < x_1 < x_2$ so the function t in increasing in $[0, x_2]$. To prove that t is decreasing when $x > x_2$ we use the same reasoning, this time with $x_1 \in (x_2, x_0)$, and prove again that $\boldsymbol{x_0} \in L_{(\boldsymbol{x_2}, \boldsymbol{x_1})}$.

Let $\tilde{L} \in \mathbb{S}^+$ stand for the line that passes through $\boldsymbol{x_0}$ and $\boldsymbol{x_1}$ and x_2 be the point at which $t(x_2) > t(x)$, $\forall x \in [0, \tilde{L}^1]$. When $x_0 < x_1 \leq x_2$, as previously seen, $t(x)$ is increasing between x_0 and x_1 and, as $x_0 < x_\alpha < x_1$, $f(\boldsymbol{x_\alpha}) = t(x_\alpha) > t(x_0) \geq \min\{f(\boldsymbol{x_0}), f(\boldsymbol{x_1})\}$; when $x_2 \leq x_0 < x_1$ then $x_0 < x_\alpha < x_1$, as t is decreasing for $x > x_2$, $t(x_\alpha) > t(x_1)$, $f(\boldsymbol{x_\alpha}) > f(\boldsymbol{x_1}) \geq \min\{f(\boldsymbol{x_0}), f(\boldsymbol{x_1})\}$; if $x_0 \leq x_2 \leq x_1$, and $x_0 < x_\alpha \leq x_2$, $t(x_\alpha) > t(x_0)$ and $f(\boldsymbol{x_\alpha}) > f(\boldsymbol{x_0}) \geq \min\{f(\boldsymbol{x_0}), f(\boldsymbol{x_1})\}$; if $x_0 \leq x_2 \leq x_1$ and $x_2 \leq x_\alpha < x_1$, $t(x_\alpha) > t(x_1)$ and $f(\boldsymbol{x_\alpha}) > f(\boldsymbol{x_1}) \geq \min\{f(\boldsymbol{x_0}), f(\boldsymbol{x_1})\}$.

We conclude that $f(\boldsymbol{x_\alpha}) = t(x_\alpha) > \min\{t(x_0), t(x_1)\} = \min\{f(\boldsymbol{x_0}), f(\boldsymbol{x_1})\}$ and as for any possibility $f(\boldsymbol{x_\alpha}) = f(\alpha\boldsymbol{x_0} + (1-\alpha)\boldsymbol{x_1}) > \min\{f(\boldsymbol{x_0}), f(\boldsymbol{x_1})\}$ the function is strictly quasiconcave. □

Theorem 3. *The choice $c(S)$ is representable and $c(S) = \arg_{\boldsymbol{x}\in S} \max x_1 x_2$*

Proof. The line $\tilde{L}$ that passes through the points $(2,0)$ and $(0,2)$ it is symmetric and due to *PO* and *Sym* $c(\tilde{L}) = (1,1)$. The line L^0 that has $c(L^0) = (z_1, z_2) = \boldsymbol{z}$ is such that $L^0 = \boldsymbol{z}\tilde{L}$ and touches $(2z_1, 0)$ and $(0, 2z_2)$, this line can be described as: $(x,y) \in L^0$ if $y = y^0(x) = 2z_2 + 2x_2/x_1 x$, for $x \in [0, 2z_1]$.

If we create a new line L^1 that passes through $c(L^0) = \boldsymbol{z}$ and $\big(\gamma(2z_1), 0\big)$, for $\gamma > 0$, then $L^1 = \boldsymbol{\alpha}L^0$, with $\boldsymbol{\alpha} = \big(\gamma, \gamma/(2\gamma-1)\big)$. Or, writing the expression for $(x,y) \in L^1$, $y = y^1(x) = \gamma/(2\gamma-1)2z_2 - 1/(2\gamma-1)(z_2/z_1)x$, for $x \in [0, \gamma(2z_1)]$.

As $c(L^0) = (z_1, z_2) \in L^1$ and $c(L^1) = (\gamma z_1, \gamma/(2\gamma-1)z_2)$ by theorem(2) we know that for any $x \in \big[\min\{z_1, \gamma z_1\}, \max\{z_1, \gamma z_1\}\big]$, $f(x, y^1(x)) > f(z_1, z_2)$. This construction of L^1 allowed us to find a set of points that are better than the initial $c(L^0)$. If we proceed the same way and construct $L^2 = \boldsymbol{\alpha}L^1$, we will find a set of points that are better than $c(L^1)$ and therefore better than $c(L^0)$. So we can find a sequence of lines $L^k = \boldsymbol{\alpha}^k L^0$ that have a part that is better than $c(L^0)$. For $\gamma > 1$ and $x \geq z_1$ we can define the function $y_\gamma(x) = y^k(x), \gamma^{k-1}z_1 \leq x < \gamma^k z_1$. So

$$y_\gamma(x) = \Big(\frac{\gamma}{2\gamma-1}\Big)^k 2z_2 - \Big(\frac{1}{2\gamma-1}\Big)^k \frac{z_2}{z_1} x \text{ for } \gamma^{k-1}z_1 \leq x < \gamma^k z_1 \tag{2}$$

Any point $(x, y_\gamma(x))$ is better than $c(L^0)$, because by theorem(2) $(x, y_\gamma(x))$ is better than $\gamma^{k-1}\boldsymbol{z}$ for $\gamma^{k-1}z_1 \leq x < \gamma^k z_1$ and by an inductive argument we know that $\gamma^{k-1}\boldsymbol{z}$ is better than $c(L^0)$. This is true whatever the value for γ so we may find $y(x) = \lim_{\gamma\downarrow 1} y_\gamma(x)$. Any point above $(x, y(x))$ is better than $\boldsymbol{z}$. As $\gamma^{k-1}z_1 \leq x < \gamma^k z_1$ we know that $\ln(x/z)/\ln\gamma \leq k < \ln(x/z)/\ln\gamma + 1$. Due to $\gamma/(2\gamma-1) < 1$ for $\gamma > 1$,

$$\Big(\frac{\gamma}{2\gamma-1}\Big)^{\frac{\ln(x/z)}{\ln\gamma}+1} < \Big(\frac{\gamma}{2\gamma-1}\Big)^k < \Big(\frac{\gamma}{2\gamma-1}\Big)^{\frac{\ln(x/z)}{\ln\gamma}} \tag{3}$$

Taking the logarithm and applying the L'Hôpital rule we get $\lim_{\gamma\downarrow 1}\big(\gamma/(2\gamma-1)\big)^{\ln(x/z)/\ln\gamma+1} =$

$\lim_{\gamma\downarrow 1}\big(\gamma/(2\gamma-1)\big)^{\ln(x/z)/\ln\gamma} = z_1/x$, and by (3) we conclude $\lim_{\gamma\downarrow 1}\big(\gamma/(2\gamma-1)\big)^k = z_1/x$. Applying the same type of calculation we get $\lim_{\gamma\downarrow 1}\big(1/(2\gamma-1)\big)^k = (z_1/x)^2$. Substituting in (3) we derive that $y(x) = \lim_{\gamma\downarrow 1} y_\gamma(x) = (z_1/x)(2z_2) - (z_1/x)^2(z_2/z_1)x = z_1z_2/x$.

We can apply the same reasoning for $\gamma < 1$ and define a function $y(x) = \lim_{\gamma\uparrow 1} y_\gamma(x)$ for $x < z_1$ with $y_\gamma(x)$ as follows

$$y_\gamma(x) = \big(\frac{\gamma}{2\gamma-1}\big)^k 2z_2 - \big(\frac{1}{2\gamma-1}\big)^k \frac{z_2}{z_1}x \text{ for } \gamma^k z_1 \leq x < \gamma^{k-1}z_1 \tag{4}$$

And, as before, it can be derived that $y(x) = z_1z_2/x$, this time for $x < z_1$. So for all points (x, y) such that $xy > z_1z_2$, $f(x, y) > f(z_1, z_2)$. Then, to chose the best option in a set S for the function f is the same as choosing the best for the function $g(z_1, z_2) = z_1z_2$ on the same set, $c(S) = \arg_{\boldsymbol{x}\in S} \max f(x_1, x_2) = \arg_{\boldsymbol{x}\in S} \max x_1x_2$. $\square$

The generalization to n players is straightforward. Let $\boldsymbol{x} \in \mathbb{R}^{n-2}$, the (generalized) Nash axioms for n players, for the compact and convex sets of the form $S_{|\boldsymbol{x}} = \{s \in \mathbb{R}^n : s_i = x_i \text{ for } i = 3, \ldots, n\}$, must be equivalent to the Nash axioms for two players. Therefore the solution $\boldsymbol{x}^* = c(S_{|\boldsymbol{x}})$ is such that $(x_1^*, x_2^*) = \arg\max_{(x_1,x_2,\boldsymbol{x})\in S_{|\boldsymbol{x}}} s_1s_2$. For a general compact and convex $S \subset \mathbb{R}^n$ if $\boldsymbol{x} = \big(c(S)_3, \ldots, c(S)_n\big)$ by *IIA*, $c(S) = c(S_{|\boldsymbol{x}})$, the bargaining solution must be such that $(c(S)_1, c(S)_2) = \arg\max_{(s_1,s_2,\boldsymbol{x})\in S} s_1s_2$. The same happens for any two players i and j, the solution must maximize x_ix_j in the section where the values for the other players are fixed at $c(S)$. Formally, with $\boldsymbol{x}^* = c(S)$, $f(\boldsymbol{x}) = \Pi_{i=1}^n x_i$, $g_i(\boldsymbol{x}) = x_i - x_i^*$ for $i \in \{1, \ldots, n\}$ and $g_{n+1}(\boldsymbol{x}) = x_n - g_S(x_1, \ldots, x_{n-1})$. $\boldsymbol{x}^*$ maximizes f on each of the manifolds defined by $\cap_{k\neq i,j} g_k^{-1}(0)$ with $i, j \in \{1, \ldots, n\}$ and $i \neq j$. Then it exists, Edwards (2012)[theorem5.8, pag.113], $\{\lambda_k^{ij}\}_{k\neq i,j}$ such that $\nabla f(\boldsymbol{x}^*) = \sum_{k\neq i,j} \lambda_k^{ij} \nabla g_k(\boldsymbol{x}^*)$. From these equations it can be derived that $f_i'/g_{S,i}' = f_j'/g_{S,j}'$ at $\boldsymbol{x}^*$ for any i, j pair. So exists a λ such that $\nabla f(\boldsymbol{x}^*) = \lambda \nabla g_S(\boldsymbol{x}^*)$, this equation is the condition for $\boldsymbol{x}^*$ to be a local extremum of f in the manifold defined by g_S, in this case it must be that $\boldsymbol{x}^*$ is a local maximand of f. We know that, Avriel, Diewert, Schaible, and Zang (2010)[Proposition 3.3, pag.58], that a local maximum of a quasi concave function is a global maximum and so $c(S) = \boldsymbol{x}^* = \arg\max_{\boldsymbol{x} \in S} f(\boldsymbol{x})$.

5 Conclusion

In this paper we developed a new method to find Nash's solution to the bargaining problem. Peters and Wakker (1991) provide the conditions for the representability of the Nash bargaining solution. Then, from the quasiconcavity and the AT Nash solution is found. The mathematical arguments used in this paper are mainly of real analysis origin and are not directly adaptable to different bargaining structures, such as for example those defined in Peters and Vermeulen (2012), Conley and Wilkie (1996) or to Kalai and Smorodinsky (1975). However, axiomatic bargaining does exhibit algebraic properties which can be explored in future research to overcome this limitation. Namely, we can regard the AT axiom as a morphism, and with the right definition of the multiplication operation on the bargaining sets, each bargaining model can then be interpreted algebraically. The study of the different axiomatic bargainings under this algebraic and more general framework will likely extend the understanding we detain of them.

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